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DEVELOPMENT CONTROL OF A MULTICOMMODITY ECONOMY BASED ON THE DYNAMICAL INPUT-OUTPUT MODEL

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In this paper the regular method for constructing the modified dynamic input-output model is proposed. The main characteristic of the approach is the inclusion of gross domestic product in the state vector. This allows to simulate the dynamics of the production sphere and the sphere of consumption. Some versions of the model that take into account various control possibilities of production and consumption processes are constructed. It is shown that the implementation of the investment programs is modeled by a linear control system of differential equations. Accounting of the effects of other macroeconomic indicators (taxes, wage rates, etc.) leads to a nonlinear control system. On the basis of the modified input-output model the scenario approach for development of investment plans and their correction is proposed. The connection of these problems with problems of construction of programmed controls and stabilization of programmed motions of dynamical systems is shown. Bibliogr. 28. Table 1.

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УПРАВЛЕНИЕ РАЗВИТИЕМ МНОГОПРОДУКТОВОЙ ЭКОНОМИКИ НА ОСНОВЕ ДИНАМИЧЕСКОЙ МОДЕЛИ «ЗАТРАТЫ-ВЫПУСК»

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В настоящей работе предложен регулярный метод построения модифицированной динамической модели межотраслевого баланса (МОБ). Основной особенностью подхода является включение валового внутреннего продукта в вектор состояния системы. Это позволяет одновременно моделировать динамику сферы производства и сферы потребления. Построены варианты модели, учитывающие различные возможности управления процессом производства и потребления. Показано, что реализация инвестиционных программ моделируется линейной управляемой системой дифференциальных уравнений. Учет влияния других макроэкономических показателей (налогов, ставок оплаты труда и др.) приводит к нелинейным управляемым системам. На основе модифицированной модели МОБ предлагается сценарный подход разработки инвестиционных планов и их коррекции.

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Показана связь этих задач с задачами построения программных управлений и стабилизации программных движений динамических систем. Библиогр. 28 назв. Табл. 1.

Ключевые слова: динамическая модель «затраты-выпуск», управление, сценарный подход.

1. Introduction. The problem statement. The country's economy at the present stage of development is a multi-sector complex with intercrossing relations. The structure of the economy sectors and their relationships are constantly changing under the influence of continuously developing and deepening processes of division and cooperation of social labor. In world practice, intersectoral balances are widely used to identify relations between economy sectors, and for the analysis of economic development.

In Russia, the statistics on changes in the coefficients of input-output tables are published annually by Rosstat. Their nomenclature is defined by the System of National Accounts (SNA). The SNA is a set of interrelated indicators that are used in the analysis of the proportions and relationships that exist in the economy. This system is standardized internationally. In 2008, the United Nations Statistical Commission adopted an updated version of the "2008 SNA", which is recommended by the Interstate Statistical Committee of the Commonwealth of Independent States (CIS) for use in the statistical services of the CIS countries [1]. The provisions of the 2008 SNA adopted for use by Rosstat to the practice since 2009.

Speaking about the relevance of this information, it should be emphasized that the main provisions of the 2008 SNA introduced a new asset – "Computer software and databases". Their costs (as well as the results of research activities) should be reflected in the documents as gross fixed capital formation. In the balance sheet for this article allocated a separate position in the fixed assets. Thus the direction of the statistical analysis of the economy at the emergence of new industries is formulated. It is an account of information technologies (IT), which are equal to the means of production. So the databases of the 2008 SNA can be attributed to the IT industry. But they can provide a profit as fixed assets, only on condition that all the information contained in them is fully used.

The basis for modeling of a multicommodity economy is the dynamical input-output model. In this model every economy sector is seen both as a consumer and a producer. At the initial stage of construction of the dynamical input-output model it is important to identify the state variables, and to develop a regular method to derive the differential equations, including the search algorithms for coefficients according to the statistical observations. All these points are the goals of this work.

The theoretical foundations of the input-output model were provided by works of Nobel laureates in Economics W. W. Leontief and L. V. Kantorovich [2–4]. Currently, the input-output model is one of the internationally acknowledged scientific instrument for analysis of regional economic and social systems, as well as macroeconomic trends in these systems. The International Input-Output Association [5], which brings together scientists concerned with the theory and practice of application of input-output models, has existed and has been actively functioning for 25 years.

In the USSR, the theoretical bases of the interbranch balance were laid in the twenties of the last century. The first balance in the history of the reporting balances of the national economy of the USSR, created in the form of an interbranch relations table, was calculated for 1923/24 financial year. But then the computing capabilities and common level of scientific research did not allow to develop this method and include it in the practice of national economic planning.

In the postwar years, due to works of Leontief the input-output model became one of the tools of rational prediction of economic growth, structural changes and employment for corporations and government services of the USA. In the 50–60s of the last century the analytical input-output method has already been used in most countries and in international calculations undertaken by UN agencies.

In the USSR, works in this direction were resumed in the early 60s under the leadership of academician V. S. Nemchinova. Experimental optimization calculations of the economy were conducted; some modifications of the input-output model (including balances of material, cost, and labor) were created. The materials of reporting balances were being published in statistical yearbooks. For the development and implementation of the input-output model in practice in 1968 the group of Soviet economists was awarded the USSR State Prize. This group included academician A. N. Efimov (head of works), E. F. Baranov, L. Ya. Berry, E. B. Ershov, F. N. Klotsvog, V. V. Kossov, L. E. Mints, S. S. Shatalin, M. R. Eydelman.

The next stage of the input-output models implementation is associated with the development of their dynamical analogs. The founder of the scientific school of strategic planning N. I. Veduta worked in the Soviet Union in this direction. He was one of the first who developed a dynamical input-output model. In his scheme for the first time balances of incomes and expenses of producers and consumers: the state, households, exporters and importers are systematically coordinated. In [6] many years of researches and practical experience of Veduta are summarized. Currently, dynamical input-output models are described and presented not only in scientific papers and monographs, but also in textbooks [7–13].

In this paper, the dynamical input-output model is considered as a tool of long-term trends analysis of an economy development and a theoretical basis for preparation of management decisions in an implementation of investment programs. A modification of input-output approach focused on the simultaneous modeling of the production sphere and consumption sphere is proposed [9]. One of the most important economic indicators – *gross domestic product (GDP)* is introduced in the model as one of the state variables. The corresponding differential equation is constructed for this variable. Control capabilities in this problem are divided into two classes: first, these are macroeconomic parameters regulated by the legislative and executive branches; secondly, different nature investments from internal reserves of corporations to credit resources. Accounting of the first factor generally leads to a nonlinear control system. In the second case (excluding the impact of the first factor) the model takes the form of a linear control system of differential equations. For the latter model in mathematical control theory there exists a wide set of well-known methods of the program controls and the synthesis of stabilizing feedbacks construction. This approach allows us to plan the scenarios of regional economy development based on investment programs for production sectors and correct their implementation in real time.

2. The main elements of an input-output table. In the input-output table n sectors of economy are presented. Each sector is both a producer and consumer of certain products or services. The input-output table consists of four matrices (quadrants).

The first quadrant of the input-output table is $(n \times n)$ -matrix of production sphere \mathbf{A}_p with elements p_{ij} (rub.) (see table) [9]. The columns of this matrix determine the internal consumption of each economy sector as a producer, i.e. the consumption of products of other industries for their production. For the analysis purposes it is necessary to bear in mind that the elements of this matrix are given by expression $p_{ij} = P_i a_{ij} I n_j$. They are a product of the price of the consumed product P_i (rub./i-unit), *technological coefficient*

a_{ij} and the annual production In_j (amount j -product p.a.) of j -sector. The value of the technological coefficient a_{ij} (amount i -product/ j -unit p.a.) determines the amount of i -sector product needed for the production unit of j -sector per year. Technological coefficients a_{ij} characterize the perfection of technologies used in each economy sector. All diagonal elements p_{jj} are the costs of each industry's own needs.

Input-output table for three-commodity economy: matrix of production sphere $\mathbf{Ap}\{p_{ij}\}$ (bln rub.), matrix of relative prices $\mathbf{R}\{r_{ij}\}$ (year)

Consumers → Producers ↓	Agriculture	Industry	Energy sector	Final consumption	Annual output
1. Agriculture Internal consumption Relative internal consumption	$p_{11}=71.8$ $r_{11}=0.25$	$p_{12}=57.7$ $r_{12}=0.2$	$p_{13}=0.0$ $r_{13}=0.0$	$Y_1=157.8$ $Yr_1=$ $=0.325$	$I_1=287.3$
2. Industry Internal consumption Relative internal consumption	$p_{21}=81.0$ $r_{21}=0.28$	$p_{22}=34.8$ $r_{22}=0.12$	$p_{23}=46.4$ $r_{23}=0.312$	$Y_2=128.2$ $Yr_2=$ $=0.264$	$I_2=290.4$
3. Energy sector Internal consumption Relative internal consumption	$p_{31}=54.8$ $r_{31}=0.19$	$p_{32}=18.4$ $r_{32}=0.063$	$p_{33}=20.6$ $r_{33}=0.138$	$Y_3=55.0$ $Yr_3=$ $=0.113$	$I_3=148.8$
Added value Relative added value	$V_1=79.7$ $Vr_1=$ $=0.277$	$V_2=179.5$ $Vr_2=$ $=0.618$	$V_3=81.8$ $Vr_3=$ $=0.55$	$V_b=145.0$ $rg=V_b/I_4=$ $=0.298$	$I_4=BBII=$ $=486.0$
Annual output	$I_1=287.3$	$I_2=290.4$	$I_3=148.8$	$I_4=BBII=$ $=486.0$	$I_{ss}=$ $=1212.5$
Compensation of employees Relative compensation	$W_1=47.7$ $Wr_1=$ $=0.166$	$W_2=108.0$ $Wr_2=$ $=0.372$	$W_3=47.8$ $Wr_3=$ $=0.321$	$W_b=93.2$ $Wr_b=$ $=0.192$	$rw=W/V=$ $=0.597$ $K_b=W_b/W=$ $=0.458$
Profit before taxes Relative profit	$Pr_1=32.0$ $Prr_1=$ $=0.111$	$Pr_2=71.5$ $Prr_2=$ $=0.246$	$Pr_3=34.0$ $Prr_3=$ $=0.228$	$Pr_b=51.8$ $Prr_b=$ $=0.107$	$Pr=189.3$
Operating costs Relative operating costs	$Pc_1=$ $=255.3$ $Rs_1=$ $=0.889$	$Pc_2=$ $=218.9$ $Rs_2=$ $=0.754$	$Pc_3=$ $=114.8$ $Rs_3=$ $=0.772$	$Pc_b=$ $=434.2$ $Rs_b=$ $=0.893$	$Pc=$ $=1023.2$ $Rs=$ $=0.844$
Profitability	$Rnt_1=$ $=0.125$	$Rnt_2=$ $=0.327$	$Rnt_3=$ $=0.296$	$Rnt_b=$ $=0.119$	$Rnt_0=$ $=0.185$

The elements sum of each column of matrix \mathbf{Ap} is equal to the value of internal consumption Pp_j in j -sector. Each row element p_{ij} of matrix \mathbf{Ap} determines the production cost, which i -sector, regarded as a producer, supplies to each j -sector per year. The sum of the elements of each i -row and the final consumption is equal to the volume of annual sales X_i of this economy sector.

The second quadrant of the input-output table (see $(n + 1)$ -th column of table) is n -dimensional column of costs of the final consumption production $\mathbf{Y} = (Y_1, \dots, Y_n)^T = (P_1Yn_1, \dots, P_nYn_n)^T$.

The third quadrant of the input-output table is $(n + 1)$ -th row \mathbf{V} . Its elements V_j are indicators of added values. Added value V_j created in every economy sector is determined by the difference between the expected value of the annual output $I_j = P_jIn_j$ in the j -sector and the value of internal consumption Pp_j , i.e. $V_j = I_j - Pp_j$. Added value V_j includes three components. The first is compensation of employees W_j . The second is value of taxes

Tx_j , which are set by the government. And the third is *the net operating surplus* Prh_j . The net operating surplus is a source of investment in the economy, and income of the employer. It is necessary to bear in mind that W_j and profit before taxes Pr_j are exogenously given parameters. Their values are set by external management (administrative) decisions based on predictive assessment of the expected market opportunities and previous experience. Finally, the sum of the cost of internal consumption and the compensation of employees is *the operating costs (the production costs)* $Pc_j = Pp_j + W_j$. Then $Pr_j = I_j - Pc_j$ or $Pr_j = V_j - W_j$.

The fourth quadrant of the input-output table located below the second is *the state budget* V_b . The state budget is formed as the sum of all taxes and other payments. At the same time V_b is an exogenous parameter, i.e. its value, on the one hand, is related to the last year period, but on the other hand, is given by the external management decisions.

Under conditions of equilibrium economy the annual sales are equal to the annual output $X_i = I_i$, and the total added value – to the total consumption of $V = Y$. In this case, the basic balance relations recorded in relative terms, take the form [9]

$$I_i = r_{i1}I_1 + \dots + r_{in}I_n + Yr_iI_{n+1}, \quad i = 1, \dots, n, \quad (1)$$

where

$$r_{ij} = \frac{p_{ij}}{I_j} = \frac{P_i a_{ij} I_{nj}}{P_j I_{nj}} = \frac{P_i a_{ij}}{P_j}$$

are elements of *the relative prices matrix* $\mathbf{R}\{r_{ij}\}$, I_{n+1} is *GDP*, $Yr_i = Y_i/I_{n+1}$ are components of final consumption normed by *GDP*.

Under the *GDP* (rub. p.a.) hereafter we mean the sum of added values V_j , created in the economy sectors and budget V_b , which is considered an added value of consumption sphere.

3. Construction of dynamical input-output model. In this section the construction method of a system of differential equations describing output changes in the economy sectors and *GDP* is suggested.

Let us consider the balance relation

$$V_j = W_j + Pr_j = W_j + Tx_j + Prh_j. \quad (2)$$

Each of the values presented in (2) is a part of the annual output in j -sector:

$$Pp_j = rp_j I_j, \quad V_j = (1 - rp_j)I_j, \quad rp_j = \sum_{i=1}^n r_{ij}, \quad (3)$$

$$W_j = rw_j V_j = rw_j(1 - rp_j)I_j, \quad Pr_j = (1 - rw_j)V_j = (1 - rw_j)(1 - rp_j)I_j.$$

In (3) rp_j is a coefficient that determines the total part of internal consumption Pp_j in the annual output I_j , rw_j is *the average rate of wages* in j -sector.

Let us introduce *the profit tax* tp and assume that it is the same for all economy sectors. Then, taking into account (3), we obtain

$$\begin{aligned} Tx_j &= tpPr_j = tp(1 - rw_j)(1 - rp_j)I_j, \\ Prh_j &= (1 - tp)Pr_j = (1 - tp)(1 - rw_j)(1 - rp_j)I_j. \end{aligned} \quad (4)$$

The next assumption is that the net operating surplus Prh_j is a source of investment $Cp_j = Prh_j$, which is used for production expansion of each economy sector. The change

of outputs $I_j(t)$ (rub./year) in time is defined as its acceleration $\dot{I}_j(t)$ (rub./year²). Investments needed to increase output (acceleration) can be considered proportional to the desired acceleration. Then the relations that determine the investment amounts in each economy sector take the following form [9]:

$$Cp_j(t) = Fe_j \dot{I}_j(t), \quad j = 1, \dots, n, \quad (5)$$

where the values Fe_j represent *capital intensities* of each sector. They are the proportionality coefficients between output accelerations and the investment amounts needed to achieve them. In fact, capital intensity characterizes a capital cost per unit of output increase per unit of time.

Further, taking into account the assumption $Cp_j = Prh_j$, formula (4) and balance relations (1), equation (5) can be written in the following form:

$$\dot{I}_j = \frac{(1 - tp)(1 - rw_j)(1 - rp_j)}{Fe_j} (r_{j1}I_1 + \dots + r_{jn}I_n + Yr_jI_{n+1}), \quad j = 1, \dots, n. \quad (6)$$

System (6) describes the dynamics of output in all sectors of the production sphere. Let us complete this system with $(n+1)$ -th equation that describes the consumption sphere, i.e. dynamics of $GDP = I_{n+1}$. For this aim we introduce the notion of a *generalized tax* rg , which determines the budget part in GDP : $V_b = rgI_{n+1}$. Using the definition of GDP and formula (3), we can write the equation describing the structure of GDP , and add it to the system (1):

$$I_{n+1} = V_1 + \dots + V_n + V_b = (1 - rp_1)I_1 + \dots + (1 - rp_n)I_n + rgI_{n+1}. \quad (7)$$

By analogy with the capital intensities of the economy sectors, let us introduce the notion of *the capital intensity of the consumption sphere* Fe_b , which determines the required amount of budget capital cost Cp_b per unit GDP increase per unit of time. Then, as in (5), we have

$$Cp_b(t) = Fe_b \dot{I}_{n+1}(t). \quad (8)$$

The GDP consists of the production costs of the consumption (budget) sphere and the budget profit, i.e. $I_{n+1} = Pc_b + Pr_b$. Let us denote the part Pc_b in GDP as rs_b , then $Pc_b = rs_b I_{n+1}$, $Pr_b = (1 - rs_b)I_{n+1}$. If we assume that all budget profit goes to the budget investments $Cp_b = Pr_b$, and consider equations (7), (8), we obtain the desired $(n + 1)$ -th differential equation

$$\dot{I}_{n+1} = \frac{1 - rs_b}{Fe_b} ((1 - rp_1)I_1 + \dots + (1 - rp_n)I_n + rgI_{n+1}). \quad (9)$$

Note that system (6), (9) is full because composed of $n + 1$ equations. This system allows to analyze the influence of important economic parameters, such as rp_j , rw_j , tp , rg , rs_b , on the process of production and consumption. Consider the phase variables vector $\mathbf{I} = (I_1, \dots, I_n, I_{n+1})^T$. Then system (6), (9) takes the form

$$\dot{\mathbf{I}} = \mathbf{D}\mathbf{I}, \quad (10)$$

where the system matrix is $\mathbf{D} = \mathbf{M}\tilde{\mathbf{R}}$, and

$$\tilde{\mathbf{R}} = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} & Yr_1 \\ r_{21} & r_{22} & \cdots & r_{2n} & Yr_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_{n1} & r_{n1} & \cdots & r_{nn} & Yr_n \\ 1 - rp_1 & 1 - rp_2 & \cdots & 1 - rp_n & rg \end{pmatrix}, \quad (11)$$

$$\mathbf{M} = \begin{pmatrix} \frac{(1-tp)(1-rw_1)(1-rp_1)}{Fe_1} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \frac{(1-tp)(1-rw_n)(1-rp_n)}{Fe_n} & 0 \\ 0 & \dots & 0 & \frac{1-rs_b}{Fe_b} \end{pmatrix}. \quad (12)$$

R e m a r k 1. The system of differential equations (6), (9) is one of the possible variants of the dynamical input-output model and its derivation represents the methodological principle of construction of such models. The analogy with the dynamics description of mechanical objects is possible here. Investments Cp_j (generally financial) are analogous to the forces applied to a body. Capital intensities Fe_j are analogous to masses, and \dot{I}_j – to accelerations. In addition, deriving the system we can express the structural elements of the balance relation (2) through other macroeconomic parameters to reflect their influence on the right side of differential equations.

To illustrate remark 1, let us introduce the coefficient kc_j defining the part of the net operating surplus that goes to the investment Pr_cj and consumption Pr_nj . Then $Pr_hj = Pr_nj + Pr_cj$ and balance (2) takes the form

$$V_j = W_j + Tx_j + Pr_nj + Pr_cj, \quad (13)$$

where, taking into account (4),

$$Pr_cj = kc_j Pr_hj \quad Pr_nj = (1 - kc_j) Pr_hj = (1 - kc_j)(1 - tp)(1 - rw_j)(1 - rp_j)I_j. \quad (14)$$

Moreover the interconnection between production costs, profit before taxes and added value in each sector was indicated above $Pc_j + Pr_j = I_j$, $Pr_j = V_j - W_j$. If we introduce *the coefficient of relative operating costs (relative production cost) rs_j* , as a part of Pc_j in output I_j , we can write the following expressions:

$$Pc_j = rs_j I_j, \quad Pr_j = (1 - rs_j)I_j. \quad (15)$$

Let us assume that the source of investment is value Pr_cj , i.e. $Cp_j = Pr_cj$. Then, using (4), (5), (13)–(15), we obtain the system similar to (6):

$$\begin{aligned} \dot{I}_j &= \frac{\alpha_j}{Fe_j} (r_{j1}I_1 + \dots + r_{jn}I_n + Yr_jI_{n+1}), \quad j = 1, \dots, n, \\ \alpha_j &= (1 - rs_j) - (tp + (1 - kc_j)(1 - tp))(1 - rw_j)(1 - rp_j). \end{aligned} \quad (16)$$

System (16), (9) is the modification of model (6), (9), as referred in remark 1. Its vector form coincides with (10). Matrix $\tilde{\mathbf{R}}$ is the same, and matrix \mathbf{M} has values α_j from equation (16) as diagonal elements in the first n rows. Note that both models describe the same process of the region's economy dynamics. The difference is that the second model is constructed with the most detailed balance relation (2) (see (13)). This leads to the possibility of analyzing the influence of a greater number of economic indicators. To the parameters group rp_j, rw_j, tp, rg, rs_b two more: kc_j and rs_j are added.

4. Possibility to control in the input-output model. In the previous section it was shown how the internal reserves of an economy in the form of the net operating surplus of the economy sectors or some its parts are transformed into investments, which initiate the positive region dynamics. It is clear that investments may be of external nature. In this case they are controls (external forces) that can change the dynamics in accordance

with pre-defined goals of development. System (10)–(12) in this case takes the form of a linear control system:

$$\dot{\mathbf{I}} = \mathbf{D}\mathbf{I} + \mathbf{Q}\mathbf{u}, \quad 0 \leq u_j \leq L_j, \quad (17)$$

where $\mathbf{u} = (u_1, \dots, u_{n+1})^T$ is a vector of controls (investments), L_j , $j = 1, \dots, n+1$, are nonnegative constants that define the natural restrictions on controls, \mathbf{Q} is diagonal matrix whose diagonal elements are zeros or units depending on which of the sectors receive access to an investment program.

The considered model has other control possibilities. We have already noted that many macroeconomic parameters are exogenous, i.e. are specified by the external management decisions. Let us show how this fact can be reflected in the proposed model.

Consider the case where the control parameter is the profit tax. Denote as tp the basic tax rate, and as u_{tp} its variation (control) satisfying the natural economic restriction: $|u_{tp}| \leq u_{tp}^*$, where u_{tp}^* is a positive constant. Then, the first n diagonal elements of matrix \mathbf{M} (see (12)) can be written as

$$(1 - tp - u_{tp})\beta_i, \quad \beta_i = \frac{(1 - rw_i)(1 - rp_i)}{Fe_i}, \quad i = 1, \dots, n.$$

In this case the modified matrix takes the form

$$\widetilde{\mathbf{M}} = \mathbf{M} + u_{tp}\mathbf{M}_0, \quad \mathbf{M}_0 = \text{diag}(-\beta_1, \dots, -\beta_n, 0), \quad (18)$$

here \mathbf{M} is matrix (12). Using representation (18), we obtain the following modification of system (17):

$$\dot{\mathbf{I}} = \mathbf{D}\mathbf{I} + \mathbf{Q}\mathbf{u} + u_{tp}\mathbf{D}_0\mathbf{I}, \quad 0 \leq u_j \leq L_j, \quad |u_{tp}| \leq u_{tp}^*. \quad (19)$$

In system (19) $\mathbf{D}_0 = \mathbf{M}_0\widetilde{\mathbf{R}}$, $j = 1, \dots, n+1$.

Consider the case where the control parameters are variations of the average rates of wages in economy sectors rw_i . Let rw_i be the notation of base rates, and their variations (controls) be denoted by u_{wi} . Let us introduce natural restrictions, reflecting the economic meaning of these control parameters $|u_{wi}| \leq u_{wi}^*$, where u_{wi}^* are positive constants. Then, the first n diagonal elements of the modified matrix (12) take the form

$$(1 - rw_i - u_{wi})\gamma_i, \quad \gamma_i = \frac{(1 - tp)(1 - rp_i)}{Fe_i}, \quad i = 1, \dots, n,$$

and the matrix itself may be written as

$$\widetilde{\mathbf{M}} = \mathbf{M} + \sum_{i=1}^n u_{wi}\mathbf{M}_i, \quad \mathbf{M}_i = \text{diag}(0, \dots, 0, -\gamma_i, 0, \dots, 0), \quad (20)$$

where \mathbf{M} is matrix (12). Using representation (20), we obtain the following modification of system (17):

$$\dot{\mathbf{I}} = \mathbf{D}\mathbf{I} + \mathbf{Q}\mathbf{u} + \sum_{i=1}^n u_{wi}\mathbf{D}_i\mathbf{I}, \quad 0 \leq u_j \leq L_j, \quad |u_{wi}| \leq u_{wi}^*. \quad (21)$$

In system (21) $\mathbf{D}_i = \mathbf{M}_i\widetilde{\mathbf{R}}$, $j = 1, \dots, n+1$, $i = 1, \dots, n$.

R e m a r k 2. Systems (19) and (21) are bilinear control systems [14–17] as their right-hand sides contain terms as a product of controlled parameters and state variables.

Consider now the most general case when the controlled parameters are both variations of average rates of wages rw_i and profit tax tp . Retaining the denotations of introduced variables write the representation of the diagonal elements of modified matrix (12):

$$(1 - tp - u_{tp})(1 - rw_i - u_{wi})\nu_i, \quad \nu_i = \frac{(1 - rp_i)}{Fe_i}, \quad i = 1, \dots, n,$$

or

$$((1 - tp)(1 - rw_i) - (1 - tp)u_{wi} - (1 - rw_i)u_{tp} + u_{tp}u_{wi})\nu_i,$$

and the matrix itself may be written as

$$\widetilde{\mathbf{M}} = \mathbf{M} + u_{tp}\widetilde{\mathbf{M}}_0 + \sum_{i=1}^n u_{wi}\widetilde{\mathbf{M}}_i + u_{tp}\mathbf{U}_w\mathbf{V}, \quad (22)$$

where \mathbf{M} is matrix (12),

$$\widetilde{\mathbf{M}}_0 = \text{diag}((rw_1 - 1)\nu_1, \dots, (rw_n - 1)\nu_n, 0), \quad \widetilde{\mathbf{M}}_i = \text{diag}(0, \dots, 0, (tp - 1)\nu_i, 0, \dots, 0),$$

$$\mathbf{U}_w = \text{diag}(u_{w1}, \dots, u_{wn}, 0), \quad \mathbf{V} = \text{diag}(\nu_1, \dots, \nu_n, 0).$$

Note that the diagonal elements of matrix \mathbf{U}_w are controlled parameters. Using representation (22), we obtain the following modification of system (17):

$$\begin{aligned} \dot{\mathbf{I}} &= \mathbf{D}\mathbf{I} + \mathbf{Q}\mathbf{u} + u_{tp}\widetilde{\mathbf{D}}_0\mathbf{I} + \sum_{i=1}^n u_{wi}\widetilde{\mathbf{D}}_i\mathbf{I} + u_{tp}\mathbf{U}_w\widetilde{\mathbf{V}}\mathbf{I}, \\ 0 &\leq u_j \leq L_j, \quad |u_{tp}| \leq u_{tp}^*, \quad |u_{wi}| \leq u_{wi}^*. \end{aligned} \quad (23)$$

In system (23) $\widetilde{\mathbf{D}}_0 = \widetilde{\mathbf{M}}_0\widetilde{\mathbf{R}}$, $\widetilde{\mathbf{D}}_i = \widetilde{\mathbf{M}}_i\widetilde{\mathbf{R}}$, $\widetilde{\mathbf{V}} = \mathbf{V}\widetilde{\mathbf{R}}$, $j = 1, \dots, n + 1$, $i = 1, \dots, n$.

System (23) is a nonlinear control system because its right-hand sides are polynomials of the second degree with respect to controlled parameters.

5. A scenario approach of realization of investment programs in a region economy. The problem of modeling and prediction of macroeconomic trends can be solved based on the dynamical input-output model (10)–(12). The most common solution algorithm for this problem consists of two parts. First, statistical estimation of all main macroeconomic indicators of the region is necessary. The final goal of this stage is the elements identification of matrix \mathbf{D} of system (10). Second, it is necessary to numerically integrate the constructed system of differential equations. An adequacy estimation of this model can be performed using statistical information on the region economy over some past years.

To solve the managing problem of investment programs we use control system (17). From an economic point of view the following problems are of interest. It is necessary to develop an investment plan for each economy sector for planned production growth. After that it is necessary to provide not only the implementation control of investment projects, but also to correct them in real time as needed on the basis of feedback principle. From the mathematical point of view the first problem is a problem of program control, and the second is a stabilization problem of a program motion of the controlled object [18, 19]. Because they are inextricably linked, first consider the essence of the scenario approach for implementation of program controls.

To adapt the mathematical control theory methods to applied problems of economy control we introduce the notion of an *investment scenario*. This term means the certainty

of the following items: 1) a planning horizon T – the time interval (years) at which investments are planned; 2) a set of time check points $t_0, \dots, t_N \in [t_0, t_0 + T]$; 3) check indicators for the phase variables in the time check points $I_{ij} = I_i(t_j)$, $i = 1, \dots, n + 1$, $j = 0, \dots, N$ (here may be conditions of a more general form); 4) a class of admissible functions describing investments $\mathbf{u}(t)$.

In work [9, see pp. 110–120] there is an example of construction of an investment scenario for an economy that aggregate up to two subjects, i.e. matrix \mathbf{D} has the dimensions (2×2) , at $T = 10$ years, two check points (initial and final), two corresponding values of the phase variables, and class of controls in the form of polynomials of time. The easiest way to classify the scenarios can be based on the criterion of fixation rigidity of its main elements. The example above refers to a *hard scenario*, as all parameters are uniquely determined. This certainty has a clear practical meaning, but on the other hand, we see no alternative solutions for the initial problem. Let us consider variants of *soft scenarios*.

Let us consider a general scheme of planning an investment program for soft scenario for a given pair of initial and final data $\mathbf{I}(t_0) = \mathbf{I}_0$, $\mathbf{I}(t_0 + T) = \mathbf{I}_1$ of the output vector. For this purpose we introduce the fundamental matrix $\mathbf{F}(t)$ of homogeneous system (10) consisting of its linearly independent solutions. Then the general solution of system (17) can be written in the Cauchy form. This allows constructing an integral equation for finding any program control on the time interval $t \in [t_0, t_0 + T]$:

$$\mathbf{I}_1 = \mathbf{F}(t_0 + T) \left(\mathbf{I}_0 + \int_{t_0}^{t_0+T} \mathbf{F}^{-1}(\tau) \mathbf{Q} \mathbf{u}(\tau) d\tau \right). \quad (24)$$

The integral equation (24) arises under any scenario of program control for system (17). A nuance is only that under hard scenarios the class of functions $\mathbf{u}(t)$ is given, and this function is substituted into (24). After that we obtain the system of algebraic equations for the undetermined coefficients. In general case (soft scenarios of planning) it can be shown [18] that any program control as the solution of integral equation (24) can be represented as

$$\mathbf{u}(t) = \mathbf{K}^T(t) \mathbf{c} + \varphi(t). \quad (25)$$

In (25) $\mathbf{K}(t) = \mathbf{F}^{-1}(t) \mathbf{Q}$, and vector \mathbf{c} is a solution of algebraic equations $\mathbf{A} \mathbf{c} = \mathbf{g}$, where

$$\mathbf{A} = \int_{t_0}^{t_0+T} \mathbf{K}(\tau) \mathbf{K}^T(\tau) d\tau, \quad \mathbf{g} = \mathbf{F}^{-1}(t_0 + T) \mathbf{I}_1 - \mathbf{I}_0,$$

and the vector function $\varphi(t)$ must satisfy the orthogonality condition

$$\int_{t_0}^{t_0+T} \mathbf{K}(\tau) \varphi(\tau) d\tau = 0.$$

A feature of program controls (25) is that they take into account the internal dynamics of the original system, because they depend on its fundamental matrix. From an economic point of view functions family (25) is the full representation of investment plans for economy sectors described by model (17).

If matrix \mathbf{A} is nonsingular and there are no additional restrictions on control resource, then control (25) formally exists for any pair of initial and final data. Such situation in mathematical control theory is called *full controllability* of system (17).

It should be noted that the implementation of continuous program controls is complicated. In economic systems this is due to the discrete nature of reporting (monthly, quarterly, or yearly). To solve the control problem in this case there exist construction algorithms for discrete analogs of program controls. The implementation of these controls means in practice that a specific (e.g., quarterly) investment plan for each economy sector was developed. Therefore, each economy subject receives the well-founded investment program. In addition, on the set of functions (25) the problem of optimizing investment plans for preassigned criteria can be considered.

To correct investment programs an additional control action is necessary. This action is called a stabilizing control. If under some scenario an investment program $\mathbf{u}_p(t)$ and the corresponding planned output $\mathbf{I}_p(t)$ are realized, then their correction is possible under the linear feedback $\mathbf{v}_s = \mathbf{L}(\mathbf{I}(t) - \mathbf{I}_p(t))$. Here, the difference vector $\mathbf{I}(t) - \mathbf{I}_p(t)$ characterizes the deviation measure of real and planned output. General algorithm for solving the stabilization problem (construction of the matrix \mathbf{L}) and various examples of its applications are presented in [18–20]. The resulting control for system (17) takes the form

$$\mathbf{u}(t) = \mathbf{u}_p(t) + \mathbf{L}(\mathbf{I}(t) - \mathbf{I}_p(t)). \quad (26)$$

This function describes a scientifically based program of investment and its correction algorithm in the implementation process for each economy sector.

6. Conclusion. Let us summarize the key points of this work and possible directions for further research. First of all the interrelation of mathematical control theory and the analysis methods of dynamical input-output models are shown. A detailed analysis of the model (17) is important. The phase variables vector of this system includes *GDP*. This evidence, in our opinion, makes the model sufficiently complete and distinguishes it from other analogs [10, 11]. In this work the regular method of matrix construction of systems of differential equations is explained in detail, so the model is suitable for various applications.

The described algorithms for constructing program and stabilizing controls are only the basic opportunities of control theory applications in economic dynamics. From a control theory point of view, the problems of constructing discrete analogs of continuous control functions are interesting, as well as parallel accounting of additional specific restrictions that abound the real economy. The above variant of stabilization (correction investment programs) implicitly assumes that the deviations are fully accessible for measurement, i.e. there is the case of full feedback. In practice this is not always possible. In this case, the methods of synthesis of special state observers may be involved [19, 20]. These methods enable us to restore the full deviation vector for use in stabilizing feedback channels. This class of problems is called the stabilization with incomplete feedback.

The problem of construction and implementation of the resulting control (26) also has prospects for expanding the application sphere. This formula illustrates the relationship between the program and a stabilizing control. Currently the multi-program approach is developing. This approach generalizes the stability problem of the program motion of the controlled object for the case of a given set of program motions. The founder of this scientific direction is V. I. Zubov [21, 22]. Some examples of formulations of multi-program problems and methods for their solutions including control problems for economic objects can be found in [15, 16, 23–28].

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